CCFU Proof 2

The Lorentzian Hyperboloid Identity

Given. The second-order linear recurrence:

$$C_2 : \quad x_{n+2} = x_{n+1} + x_n.$$

Assume $x_n \neq 0$. Let $R = x_{n+1}/x_n \in \mathbb{R}$.

Define.

$$U_1 = \frac{2R-1}{\sqrt{5}}, \qquad U_2 = \frac{R+2}{\sqrt{5}}, \qquad V = R.$$

Claim.

$$U_1^2 + U_2^2 - V^2 = 1.$$

Proof.

$$\begin{aligned} U_1^2 + U_2^2 &= \frac{(2R-1)^2 + (R+2)^2}{5} \\ &= \frac{4R^2 - 4R + 1 + R^2 + 4R + 4}{5} \\ &= \frac{5R^2 + 5}{5} \\ &= R^2 + 1. \end{aligned}$$

Therefore:

$$U_1^2 + U_2^2 - V^2 = R^2 + 1 - R^2 = 1. \qquad \blacksquare$$

Note. This is a one-parameter Lorentzian hyperboloid identity, not a full H^2 embedding with metric and curvature. For every real R, the point (U_1, U_2, V) lies on the unit hyperboloid $U_1^2 + U_2^2 - V^2 = 1$ in signature $(2,1)$. In the positive C_2 ratio dynamics, $R > 0$. The normalization $\sqrt{5}$ is the C_2 discriminant scale. No free parameters are chosen.

Corollary — C_2 defines a null ruling. Write

$$(U_1,\, U_2,\, V) \;=\; P + R\,D,$$

where

$$P = \left(-\tfrac{1}{\sqrt{5}},\, \tfrac{2}{\sqrt{5}},\, 0\right), \qquad D = \left(\tfrac{2}{\sqrt{5}},\, \tfrac{1}{\sqrt{5}},\, 1\right).$$

Under the Lorentzian form $\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3$:

$$\langle P, P\rangle = 1, \qquad \langle D, D\rangle = 0, \qquad \langle P, D\rangle = 0.$$

Therefore $\langle P + RD,\, P + RD\rangle = 1$ for all $R \in \mathbb{R}$. The direction D is null; the C_2 ratio curve is a null ruling on the Lorentzian hyperboloid. $\blacksquare$